Strict diagonal dominance example, in which we solve for inequalities of nonlinear functions.

Under what conditions will the following matrices be strictly diagonally dominant? Your answer must be in the form of a list of inequalities for α and β (i.e., an inequalities in terms of β^2 or β^3 *are not allowed*). In other words, you must write inequalities in the form $a_i < \alpha < b_i$ and $c_i < \beta < d_i$ that result in strict diagonal dominance for the following matrices, where a_i, b_i, c_i, d_i are constants you must find. Note that there may be multiple inequalities that α and β must satisfy, and you must find all of them.

$$\begin{bmatrix} \alpha & 4 & 10 \\ \beta^2 & 5 & \beta^4 \\ -1 & 0 & 2 \end{bmatrix}$$

The definition of strict diagonal dominance is that for any given row, the magnitude of the diagonal element is greater than the sums of magnitudes of all other elements:

$$|a_{ii}| \geq \sum_{j \neq i} |a_{ij}| \quad \text{for all } i.$$

The third row is clearly satisfied: $2 > |-1| + 0$.

The first row of this matrix is slightly less trivial, but still easy:

$$\begin{aligned} |\alpha| &> 4 + 10 \\ |\alpha| &> 14 \\ \implies -\infty < \alpha &< -14, \quad \text{and} \\ 14 < \alpha &< \infty \end{aligned}$$

Next, we attempt the second row:

$$\begin{aligned} \beta^4 + \beta^2 &< 5 \\ \gamma = \beta^2 \implies \gamma^2 + \gamma - 5 &< 0 \end{aligned}$$

The boundaries where this inequality is satisfied is given by the points where $\gamma^2+\gamma-5=0$:

$$\begin{aligned}\gamma^2+\gamma-5 &= 0\\ \implies \gamma &= \frac{-1\pm\sqrt{1+4*5}}{2}\\ &= \frac{-1\pm\sqrt{21}}{2}\end{aligned}$$

Check $\gamma=0$: satisfies the inequality, so

$$\begin{aligned}\frac{-1-\sqrt{21}}{2} &< \gamma < \frac{-1+\sqrt{21}}{2}\\ \frac{-1-\sqrt{21}}{2} &< \beta^2 < \frac{-1+\sqrt{21}}{2},\end{aligned}$$

which implies that $\beta^4+\beta^2-5=0$ at four points:

$$\beta = \pm\sqrt{\frac{-1\pm\sqrt{21}}{2}}$$

We demand β to be real, and $-1-\sqrt{21}<0$, so there are only two real roots of the equation:

$$\beta = \pm\sqrt{\frac{-1+\sqrt{21}}{2}}$$

Thus there are a total of three possible intervals where the inequality $\beta^4+\beta^2-5<0$ might be satisfied:

$$\begin{aligned}\sqrt{\frac{-1+\sqrt{21}}{2}} &< \beta < \infty\\ -\sqrt{\frac{-1+\sqrt{21}}{2}} &< \beta < \sqrt{\frac{-1+\sqrt{21}}{2}}\\ -\infty &< \beta < \sqrt{\frac{-1+\sqrt{21}}{2}}\end{aligned}$$

but which intervals actually work? Choose the limit where β is large and either positive or negative. Then $\beta^4+\beta^2-5$ is clearly greater than zero, which violates the inequality. Next choose the limit where β is zero: $\beta^4+\beta^2-5=-5<0$, which satisfies the inequality. Thus we find

$$-\sqrt{\frac{-1+\sqrt{21}}{2}} < \beta < \sqrt{\frac{-1+\sqrt{21}}{2}}$$